A Measure & Conquer Approach for the Analysis of Exact Algorithms

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The Importance of Being Tight

- (Accurately) measuring the size of relevant quantities is a crucial step in science and engineering

- Computer science, and in particular algorithm design, is not an exception

- Tight measures of (worst-case) time/space complexities, approximation ratios etc. are crucial to understand how good an algorithm is, and whether there is room for improvement

The Importance of Being Tight

- Tight bounds sometimes are shown years after the design of an algorithm

- Still, for several poly-time algorithms we are able to provide tight running time bounds

EG: The worst-case running time of `MergeSort` is $\Theta(n \log n)$

- Similarly, we have tight approximation bounds for many approximation algorithms

EG: The approximation ratio of the classical primal-dual algorithm for Steiner forest is exactly 2

The Importance of Being Tight

- The overall situation for exact (exp-time) algorithms for NP-hard problems is much worse

- Typically, tight time bounds are known only for trivial or almost trivial (enumerative) algorithms

- Nonetheless, most of the research in this field was devoted to the design of better algorithms, not of better analytical tools

$\Rightarrow$ The aim of this talk is introducing a non-standard analytical tool, sometimes named *Measure & Conquer*, which leads to much tighter (though possibly non-tight) running time bounds for branch & reduce exact algorithms

Exact Algorithms

Exact Algorithms

- The aim of exact algorithms is solving NP-hard problems exactly with the smallest possible (exponential) running time
- Exact algorithms are interesting for several reasons
  - ◇ Need for exact solutions (e.g. decision problems)
  - ◇ Reducing the running time from, say, $O(2^n)$ to $O(1.41^n)$ roughly doubles the size of the instances solvable within a given (large) time bound. This can't be achieved using faster computers!!
  - ◇ Classical approaches (heuristics, approximation algorithms, parameterized algorithms...) have limits and drawbacks (no guaranty, hardness of approximation, $W[1]$-completeness...)
  - ◇ New combinatorial and algorithmic challenges

Branch & Reduce Algorithms

- The most common exact algorithms are based on the *branch & reduce* paradigm

- The idea is to apply some *reduction rules* to reduce the size of the problem, and then branch on two or more subproblems which are solved recursively according to some *branching rules*

- The analysis of such recursive algorithms is typically based on the *bounded search tree* technique: a *measure* of the size of the subproblems is defined. This measure is used to lower bound the *progress* made by the algorithm at each branching step.

- Though these algorithms are often very complicated, measures used in their analysis are usually trivial (e.g., number of nodes or edges in the graph).

Bounded Search Trees

- Let $P(n)$ be the number of base instances generated to solve a problem of size $n \geq 0$

- Suppose, as it is usual the case, that the application of reduction and branching rules takes polynomial time (in n). Assume also that the branching depth is bounded by a polynomial

- Then the running time of the algorithm is $O(P(n)n^{O(1)}) = O^*(P(n))$
  - $O^*()$ suppresses polynomial factors

- It is possible to show by induction that $P(n) \leq \lambda^n$ for a proper constant $\lambda > 1$

Bounded Search Trees

- Consider a branching/reduction rule b which generates $h(b) \geq 1$ subproblems. Let $n - \delta_j^b$ be the size of the j-th subproblem
  - ◇ It must be $\delta_j^b \geq 0$ (indeed $\delta_j^b > 0$ for $h(b) > 1$)
  - ◇ $(\delta_1^b, \ldots, \delta_{h(b)}^b)$ is the *branching vector*
- We obtain the following inequalities

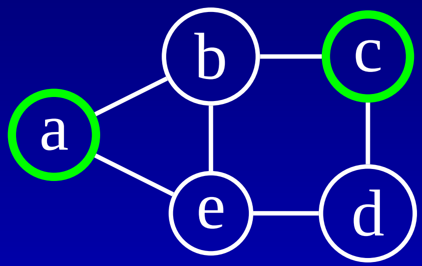
$$P(n) \leq \sum_{j=1}^{h(b)} P(n - \delta_j^b) \leq \sum_{j=1}^{h(b)} \lambda^{n-\delta_j^b} \leq \lambda^n \;\Rightarrow\; f^b(\lambda) := 1 - \sum_{j=1}^{h(b)} \lambda^{-\delta_j^b} \leq 0$$

- This gives a lower bound $\lambda \geq \lambda^b$, where λ^b is the unique positive root of $f^b(\cdot)$ (*branching factor*).
- We can conclude that $\lambda = \max_b\{\lambda^b\}$

The Independent Set Problem

Independent Set

Def: Given $G = (V, E)$, the *maximum independent set* problem (MIS) is to determine the maximum cardinality $\alpha(G)$ of a subset of pairwise non-adjacent nodes (*independent set*)



$$\alpha(G) = 2$$

Known Results

- NP-hard [Karp'72]
- Not approximable within $O(n^{1-\epsilon})$ unless $P = NP$ [Zucherman'06]
- $W[1]$-complete [Downey&Fellows'95].
- No exact $O(\lambda^{o(n)})$ algorithm unless SNP⊆SUBEXP [Impagliazzo,Paturi,Zane'01]

$\Rightarrow$ The best we can hope for is a $O(\lambda^n)$ exact algorithm for some small constant $\lambda \in (1, 2]$.

Known Results

- $O(1.261^n)$ poly-space [Tarjan&Trojanowski'77]
- $O(1.235^n)$ poly-space [Jian'86]
- $O(1.228^n)$ poly-space, $O(1.211^n)$ exp-space [Robson'86]
- better results for sparse graphs [Beigel'99, Chen,Kanj&Xia'03]

- Thanks to Measure & Conquer, a *much simpler* poly-space algorithm ($\simeq 10$ lines of pseudo-code against $\simeq 100$ lines in [Robson'86]) is shown to have time complexity $O(1.221^n)$ [Fomin, Grandoni, Kratsch'06]

$\Rightarrow$ We will consider a similar algorithm, and analyze it in a similar (but simplified) way

Reduction Rules

- Let us introduce a few standard reduction rules for MIS
  - connected components
  - domination
  - folding
  - mirroring
  - ...
- We will use only folding, but in the exercises the other rules might turn to be useful

Connected components

Lem 1: Given a graph G with connected components $G_1, \ldots, G_h$,

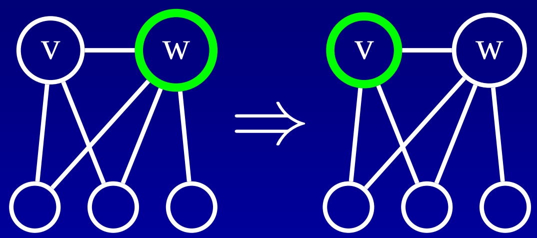
$$\alpha(G) = \sum_i \alpha(G_i)$$

Rem: One can solve the problems induced by the G_i's independently

Domination

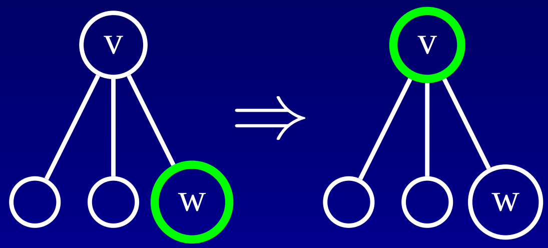
Lem 2: If there are two nodes v and w such that $N[v] \subseteq N[w]$, there is a maximum independent set which does not contain w

- $N[x] = N(x) \cup \{x\}$



Domination

Lem 3: For every node v, there is a maximum independent set which either contains v or at least two nodes in $N(v)$.

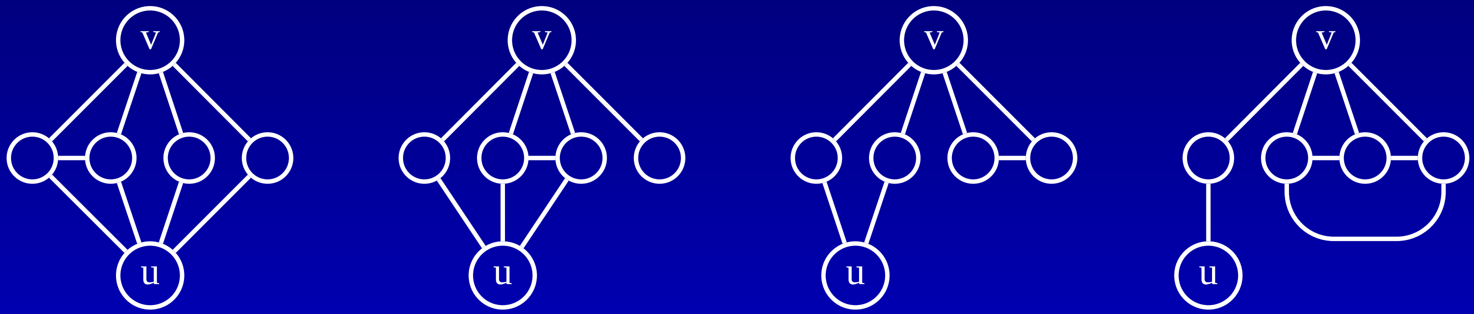


Exr 1: Prove Lemmas 1, 2, and 3

Mirroring

Def: A *mirror* of a node v is a node $u \in N^2(v)$ such that $N(v) - N(u)$ is a (possibly empty) clique

- $N^2(v)$ are the nodes at distance 2 from v
- mirrors of v are denoted by $M(v)$



Mirroring

Lem 4: For any node v,
$\alpha(G) = \max\{\alpha(G - v - M(v)), \alpha(G - N[v])\}$

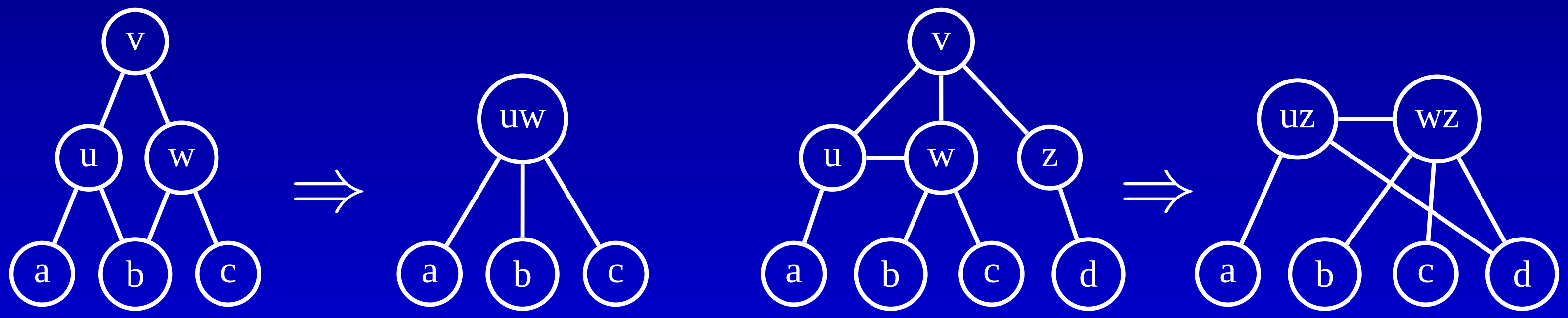
Exr: Prove Lem 4 (Hint: use Lem 3)

Folding

Def: Given a node v with no anti-triangle in $N(v)$, *folding* v means

- replacing $N[v]$ with a clique containing one node uw for each anti-edge uw of $N(v)$;
- adding edges between each uw and N(u)∪ N(w)-N[v].

◇ we use G_v to denote the graph after folding



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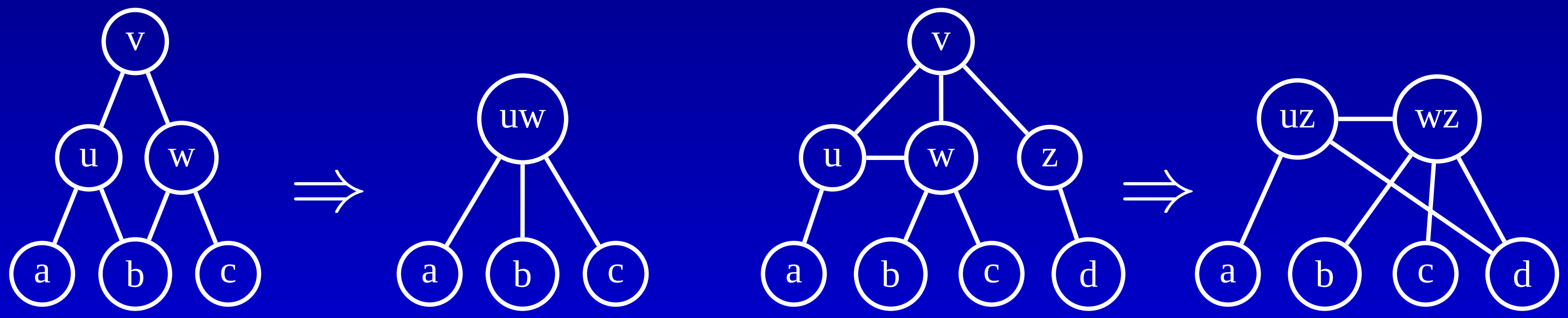

Rem 1: Folding can increase the number of nodes!

Folding

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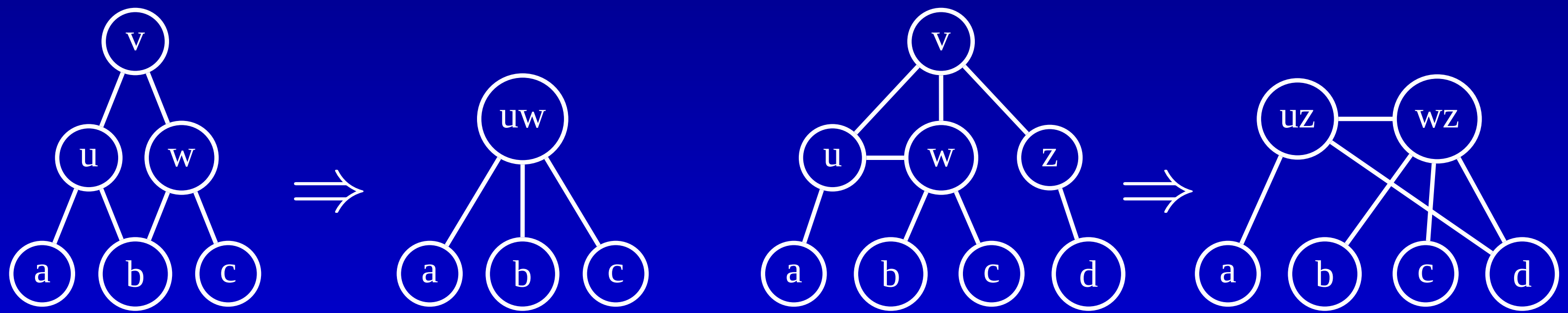
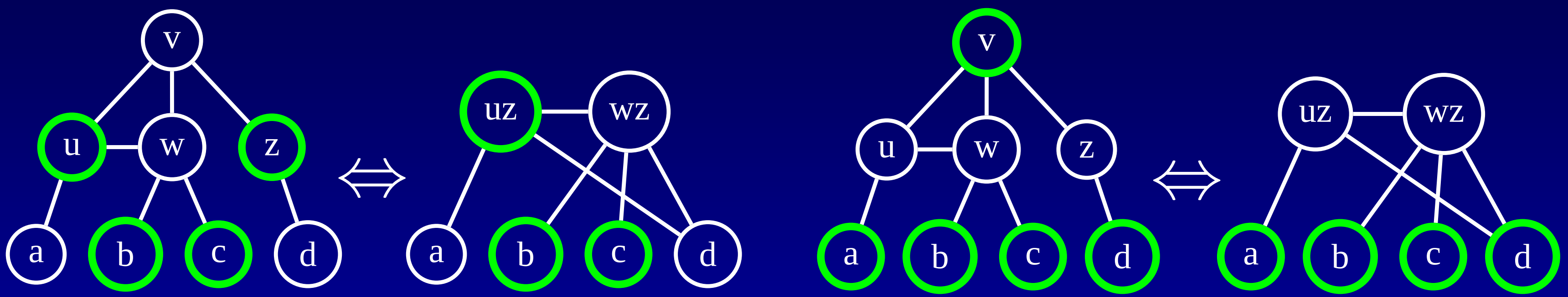


Rem 2: Nodes of degree ≤ 2 are always *foldable*

Folding

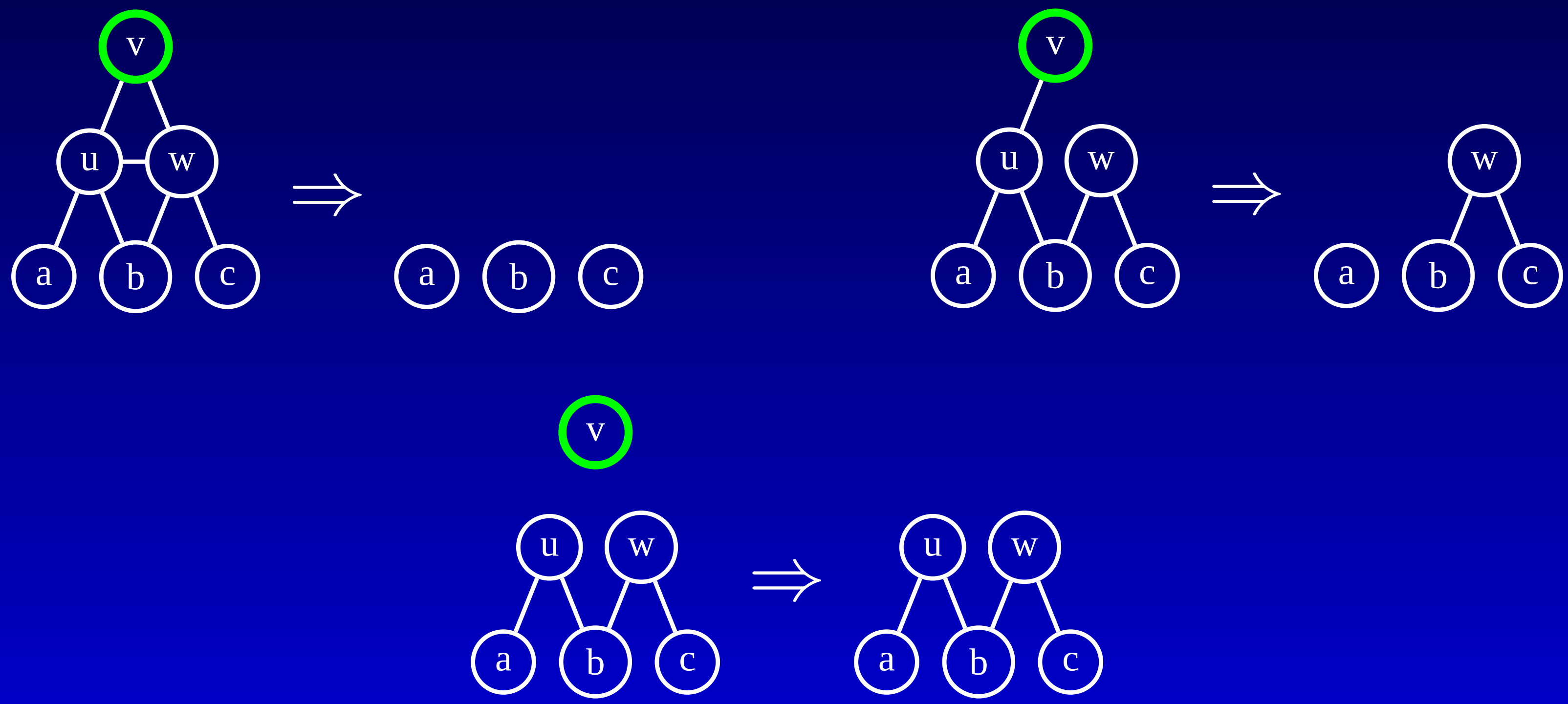
Lem 5: For a foldable node v, $\alpha(G) = 1 + \alpha(G_v)$



Exr 3: Prove Lem 5 (Hint: use Lem 3)

Folding

Rem: Lem 5 includes a few standard reductions as special cases



A Simple MIS Algorithm

int $\texttt{mis}(G)$ {

  if $(G = \emptyset)$ **return** 0; *//Base case*

  //Folding

  Take v of minimum degree;

  if $(d(v) \leq 2)$ **return** $1 + \texttt{mis}(G_v)$;

  //"Greedy" branching

  Take v of maximum degree;

  return $\max\{\, \texttt{mis}(G - v), 1 + \texttt{mis}(G - N[v])\,\}$;

}

Standard Analysis of `mis`

Thr: Algorithm `mis` solves MIS in $O^*(1.33^n)$ time

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Thr: Algorithm `mis` solves MIS in $O^*(1.33^n)$ time

Prf:

- Let $P(n)$ be the number of base instances generated by the algorithm. We will show by induction that $P(n) \leq \lambda^n$ for $\lambda < 1.33$
- In the base case $P(0) = 1 \leq \lambda^0$
- When the algorithm folds a node, the number of nodes decreases by at least one

$$P(n) \leq P(n-1) \leq \lambda^{n-1} \leq \lambda^n$$

Standard Analysis of `mis`

Thr: Algorithm `mis` solves MIS in $O^*(1.33^n)$ time

Prf:

- When the algorithm branches at a node v with $d(v) \geq 4$, in one subproblem it removes 1 node (i.e. v), and in the other it removes $1 + d(v) \geq 5$ nodes (i.e. $N[v]$):

$$\begin{aligned} P(n) &\leq P(n-1) + P(n-5) \\ &\leq \lambda^{n-1} + \lambda^{n-5} \leq \lambda^n \qquad\qquad (\lambda \geq 1.32\ldots) \end{aligned}$$

Standard Analysis of `mis`

Thr: Algorithm `mis` solves MIS in $O^*(1.33^n)$ time

Prf:

- Otherwise, the algorithm branches at a node v of degree exactly 3, hence removing either 1 or 4 nodes. However, in the first case a node of degree 2 is folded afterwards, with the removal of at least 2 more nodes

$$\begin{aligned} P(n) &\leq P(n-3) + P(n-4) \\ &\leq \lambda^{n-3} + \lambda^{n-4} \leq \lambda^n \qquad (\lambda \geq 1.22\ldots) \end{aligned}$$

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Rem: This is the best one can get with a standard analysis

Measure & Conquer

Measure & Conquer

- The classical approach to *improve* on `mis` would be designing refined branching and reduction rules. In particular, one tries to improve on the *tight* recurrences. The analysis is then performed in a similar fashion

- In the standard analysis, n is both the measure used in the analysis and the quantity in terms of which the final time bound is expressed

- However, one is free to use any, possibly sophisticated, measure m in the analysis, provided that $m \leq f(n)$ for some known function f

- This way, one achieves a time bound of the kind $O^*(\lambda^m) = O^*(\lambda^{f(n)})$, which is in the desired form

Measure & Conquer

- The idea behind Measure & Conquer is focusing on the choice of the measure

- In fact, a more sophisticated measure may capture phenomena which standard measures are not able to exploit, and hence lead to a tighter analysis of a *given* algorithm

- We next show how to get a much better time bound for `mis` thanks to a better measure of subproblems size (without changing the algorithm!)

- We will start by introducing an alternative, simple, measure. This measure does not immediately give a better time bound, but it will be a good starting point to define a really better measure

An Alternative Measure

- Nodes of degree ≤ 2 can be removed without branching
- Hence they do not really contribute to the *size* of the problem
- For example, if the maximum degree is 2, then `mis` solves the problem in polynomial time!

Idea: define the size of the problem to be the number of nodes of degree at least 3

An Alternative Measure

Def: Let n_i be the number of nodes of degree i, and $n_{\geq i} = \sum_{j \geq i} n_j$

• We define the size of the problem to be $m = n_{\geq 3}$ (rather than $m = n$)

Rem: $m = n_{\geq 3} \leq n$. Hence, if we prove a running time bound in $O^*(\lambda^m)$, we immediately get a $O^*(\lambda^n)$ time bound

An Alternative Analysis

Thr: Algorithm `mis` solves MIS in $O^*(1.33^n)$ time

An Alternative Analysis

Thr: Algorithm `mis` solves MIS in $O^*(1.33^n)$ time

(Alternative) Prf:

- Let us define G a base instance if the maximum degree in G is 2 (which implies $m = n_{\geq 3} = 0$)

- Let moreover $P(m)$ be the number of base instances generated by the algorithm to solve an instance of size m

- By the usual argument the running time is $O^*(P(m))$. We prove by induction that $P(m) \leq \lambda^m$ for $\lambda < 1.33$

An Alternative Analysis

Thr: Algorithm `mis` solves MIS in $O^*(1.33^n)$ time

(Alternative) Prf:

- In the base case $m = 0$. Thus

$$P(0) = 1 \leq \lambda^0$$

- Let m' be the size of the problem after folding a node v. It is sufficient to show that $m' \leq m$, from which

$$P(m) \leq P(m') \leq \lambda^{m'} \leq \lambda^m$$

- This condition trivially holds when folding only removes nodes

An Alternative Analysis

Thr: Algorithm `mis` solves MIS in $O^*(1.33^n)$ time

(Alternative) Prf:

- In the remaining case $N(v) = \{u, w\}$ with $uw \notin E$. In this case we remove $\{v, u, w\}$ and add a node uw with $d(uw) \leq d(u) + d(w) - 2$. By case analysis $m' \leq m$ also in this case

$d(u)$	$d(w)$	$d(uw)$	m'
2	2	2	m
2	≥ 3	≥ 3	$m - 1 + 1$
≥ 3	≥ 3	≥ 4	$m - 2 + 1$

An Alternative Analysis

Thr: Algorithm `mis` solves MIS in $O^*(1.33^n)$ time

(Alternative) Prf:

- Suppose now that we branch at a node v with $d(v) \geq 4$. Note that all the nodes of the graph have degree ≥ 3 (since we do not fold). For $t_3 = |\{u \in N(v) : d(u) = 3\}|$,

$$\begin{aligned} P(m) &\leq P(m-1-t_3) + P(m-1-d(v)) \\ &\leq P(m-1) + P(m-5) \leq \lambda^{m-1} + \lambda^{m-5} \leq \lambda^m \quad (\lambda \geq 1.32\ldots) \end{aligned}$$

- Eventually, consider branching at v, $d(v) = 3$. In this case we remove either 1 or 4 nodes of degree 3. However, in the first case the degree of the 3 neighbors of v drops from 3 to 2, with a consequent further reduction of the size by 3

$$P(m) \leq P(m-4) + P(m-4) \leq \lambda^{m-4} + \lambda^{m-4} \leq \lambda^m \quad (\lambda \geq 1.18\ldots)$$

A Better Measure

- When we branch at a node of large degree, we decrement by 1 the degree of many other nodes
- This is beneficial on long term, since we can remove nodes of degree ≤ 2 without branching
- We are not exploiting this fact in the current analysis

Idea: assign a larger *weight* $\omega_i \leq 1$ to nodes of larger degree i, and let the size of the problem be the sum of node weights. This way, when the degree of a node decreases, the size of the problem decreases as well

A Better Measure

Def:

- for a constant $\omega \in (0, 1]$ to be fixed later,

$$\omega_i = \begin{cases} 0 & \text{if } i \le 2; \\ \omega & \text{if } i = 3; \\ 1 & \text{otherwise.} \end{cases}$$

- Let $\omega(v) = \omega_{d(v)}$
- the size $m = m(G)$ of G is

$$m = \sum_{v \in V(G)} \omega(v) = \omega \cdot n_3 + n_{\ge 4}$$

A Better Analysis

Thr: Algorithm `mis` solves MIS in $O^*(1.29^n)$ time

Prf:

- With the usual notation, let us show that $P(m) \leq \lambda^m$ for $\lambda < 1.29$

- In the base case $m = 0$, $P(0) = 1 \leq \lambda^0$

- In case of folding, let m' be the size of the subproblem. it is sufficient to show that $m' \leq m$. Then

$$P(m) \leq P(m') \leq \lambda^{m'} \leq \lambda^m$$

- This condition is satisfied when nodes are only removed (being the weight increasing with the degree)

A Better Analysis

Thr: Algorithm `mis` solves MIS in $O^*(1.29^n)$ time

Prf:

- The unique remaining case is that $N(v) = \{u, w\}$, with u and w not adjacent. In this case we remove $\{v, u, w\}$, and add a node uw with $d(uw) \leq d(u) + d(w) - 2$. Hence it is sufficient to show that

$$\omega(v) + \omega(u) + \omega(w) - \omega(sw) = \omega(u) + \omega(w) - \omega(uw) \geq 0$$

A Better Analysis

Thr: Algorithm `mis` solves MIS in $O^*(1.29^n)$ time

Prf:

- By a simple case analysis

$d(u)$	$d(w)$	$d(uw)$	$\omega(u) + \omega(w) - \omega(uw) \geq 0$
2	2	2	$0 + 0 - 0 \geq 0$
2	3	3	$0 + \omega - \omega \geq 0$
2	≥ 4	≥ 4	$0 + 1 - 1 \geq 0$
3	3	4	$\omega + \omega - 1 \geq 0$
3	≥ 4	≥ 4	$\omega + 1 - 1 \geq 0$
≥ 4	≥ 4	≥ 4	$1 + 1 - 1 \geq 0$

- We can conclude that $\omega \geq \frac{1}{2}$ (new constraint on the weights!)

A Better Analysis

Thr: Algorithm `mis` solves MIS in $O^*(1.29^n)$ time

Prf:

- Consider now branching at a node v, $d(v) \geq 5$. Let d_i be the degree of the ith neighbor of v (which thus has weight ω_{d_i}). Then

$$P(m) \leq P(m - \omega_{d(v)} - \sum_i (\omega_{d_i} - \omega_{d_i - 1})) + P(m - \omega_{d(v)} - \sum_i \omega_{d_i})$$

$$\leq P(m - 1 - \sum_{i=1}^{5} (\omega_{d_i} - \omega_{d_i - 1})) + P(s - 1 - \sum_{i=1}^{5} \omega_{d_i})$$

- Observe that we can replace $d_i \geq 6$ with $d_i = 5$. In fact in both cases $\omega_{d_i} = 1$ and $\omega_{d_i} - \omega_{d_i - 1} = 0$. Hence we can assume $d_i \in \{3, 4, 5\}$ (finite number of recurrences!!!)

A Better Analysis

Thr: Algorithm `mis` solves MIS in $O^*(1.29^n)$ time

Prf:

- By case enumeration

$$P(m) \leq \begin{cases} P(m - 1 - 5\cdot\omega - 0\cdot(1\text{-}\omega) - 0\cdot 0) + P(m - 1 - 5\cdot\omega - 0\cdot 1 - 0\cdot 1) \\ P(m - 1 - 4\cdot\omega - 1\cdot(1\text{-}\omega) - 0\cdot 0) + P(m - 1 - 4\cdot\omega - 1\cdot 1 - 0\cdot 1) \\ P(m - 1 - 4\cdot\omega - 0\cdot(1\text{-}\omega) - 1\cdot 0) + P(m - 1 - 4\cdot\omega - 0\cdot 1 - 1\cdot 1) \\ P(m - 1 - 3\cdot\omega - 2\cdot(1\text{-}\omega) - 0\cdot 0) + P(m - 1 - 3\cdot\omega - 2\cdot 1 - 0\cdot 1) \\ \quad\ldots \\ P(m - 1 - 0\cdot\omega - 0\cdot(1\text{-}\omega) - 5\cdot 0) + P(m - 1 - 0\cdot\omega - 0\cdot 1 - 5\cdot 1) \end{cases}$$

A Better Analysis

Thr: Algorithm `mis` solves MIS in $O^*(1.29^n)$ time

Prf:

- Consider now branching at a node v, $d(v) = 4$. By a similar argument (but with $d_i \in \{3, 4\}$)

$$P(m) \leq \begin{cases} P(m - 1 - 4\cdot\omega - 0\cdot(1-\omega)) + P(m - 1 - 4\cdot\omega - 0\cdot 1) \\ P(m - 1 - 3\cdot\omega - 1\cdot(1-\omega)) + P(m - 1 - 3\cdot\omega - 1\cdot 1) \\ P(m - 1 - 2\cdot\omega - 2\cdot(1-\omega)) + P(m - 1 - 2\cdot\omega - 2\cdot 1) \\ P(m - 1 - 1\cdot\omega - 3\cdot(1-\omega)) + P(m - 1 - 1\cdot\omega - 3\cdot 1) \\ P(m - 1 - 0\cdot\omega - 4\cdot(1-\omega)) + P(m - 1 - 0\cdot\omega - 4\cdot 1) \end{cases}$$

A Better Analysis

Thr: Algorithm `mis` solves MIS in $O^*(1.29^n)$ time

Prf:

• Consider eventually branching at a node v, $d(v) = 3$. By an analogous argument (but with $\omega(v) = \omega_3 = \omega$ and $d_i = 3$)

$$P(m) \leq P(m - \omega - 3\,\omega) + P(m - \omega - 3\,\omega)$$

• For every $\omega \in [0.5, 1]$, the set of recurrences above provides an upper bound $\lambda(\omega)$ on λ. Our goal is minimizing $\lambda(\omega)$ (hence getting a better time bound)

• Via exhaustive (grid) enumeration, we obtained $\omega = 0.7$ which gives $\lambda(\omega) < 1.29$

An Even Better Measure

- We can extend the previous approach to larger degrees

$$\omega_i = \begin{cases} 0 & \text{if } i \leq 2; \\ \omega & \text{if } i = 3; \\ \omega' & \text{if } i = 4; \\ 1 & \text{otherwise.} \end{cases}$$

where $0 < \omega \leq \omega' \leq 1$

Thr 3: Algorithm `mis` solves MIS in $O^*(1.26^n)$ time

Exercises

Exr 5: Prove Thr 3 (Hint: $\omega = 0.750$, $\omega' = 0.951$)

Exr 6: What do you expect that would happen if we added one extra weight $\omega_5 = \omega''$? Can you guess any pattern?

Exr 7*: Design a better algorithm for MIS, using possibly the other mentioned reduction rules. Analyze your algorithm in the standard way and via Measure & Conquer

Exr 8*: Can you imagine an alternative, promising measure for MIS?

Quasiconvex Analysis of Backtracking Algorithms

Optimal Weights Computation

- When the number of distinct weights grows, an exhaustive exploration might be too slow
- We next describe a general tool to perform this computation in an (exponentially) faster way

Multivariate Recurrences

- Consider a collection of integral *measures* $m_1, \ldots, m_d$, describing different aspects of the size of the problem considered

EG: In the analysis of `mis` we used $m_1 = n_3$ and $m_2 = n_{\geq 4}$

- These measure naturally induce a set of multivariate recurrence of the following kind for each branching b

$$P(m_1, \ldots, m_d) \leq P(m_1 - \delta^b_{1,1}, \ldots, m_d - \delta^b_{d,1}) + \ldots$$
$$+ P(m_1 - \delta^b_{1,h(b)}, \ldots, m_d - \delta^b_{d,h(b)})$$

Rem: some of the $\delta^b_{i,j}$ might be negative. For example, deleting one edge incident to a node of degree 4, we decrease $n_{\geq 4}$ but increase n_3

Multivariate Recurrences

- Solving multivariate recurrences is typically rather complicated

- A common alternative is turning them into univariate recurrences by considering a linear combination of the measures (*aggregated measure*)

$$m(\alpha) = \alpha_1\, m_1 + \ldots + \alpha_d\, m_d$$

- The *weights* α_i must satisfy the condition $\delta_j^b := \sum_i \alpha_i\, \delta_{i,j}^b > 0$, i.e. $m(\alpha)$ decreases in each subproblem (we allow ≥ 0 for $h = 1$)

EG: In the analysis of `mis` we used $\alpha_1 = \omega$ and $\alpha_2 = 1$. The condition is satisfied for every $\omega \in [0.5, 1]$

Multivariate Recurrences

- The resulting set of univariate recurrences can be solved in the standard way (for fixed weights)

- In particular, for each branching b we compute the (unique) positive root $\lambda^b(\alpha)$ of

$$f^b(\lambda, \alpha) := 1 - \sum_j \lambda^{-\sum_i \alpha_i \delta^b_{i,j}}$$

- This gives a running time bound of the kind $O^*(\lambda(\alpha)^{\sum_i \alpha_i m_i})$ where

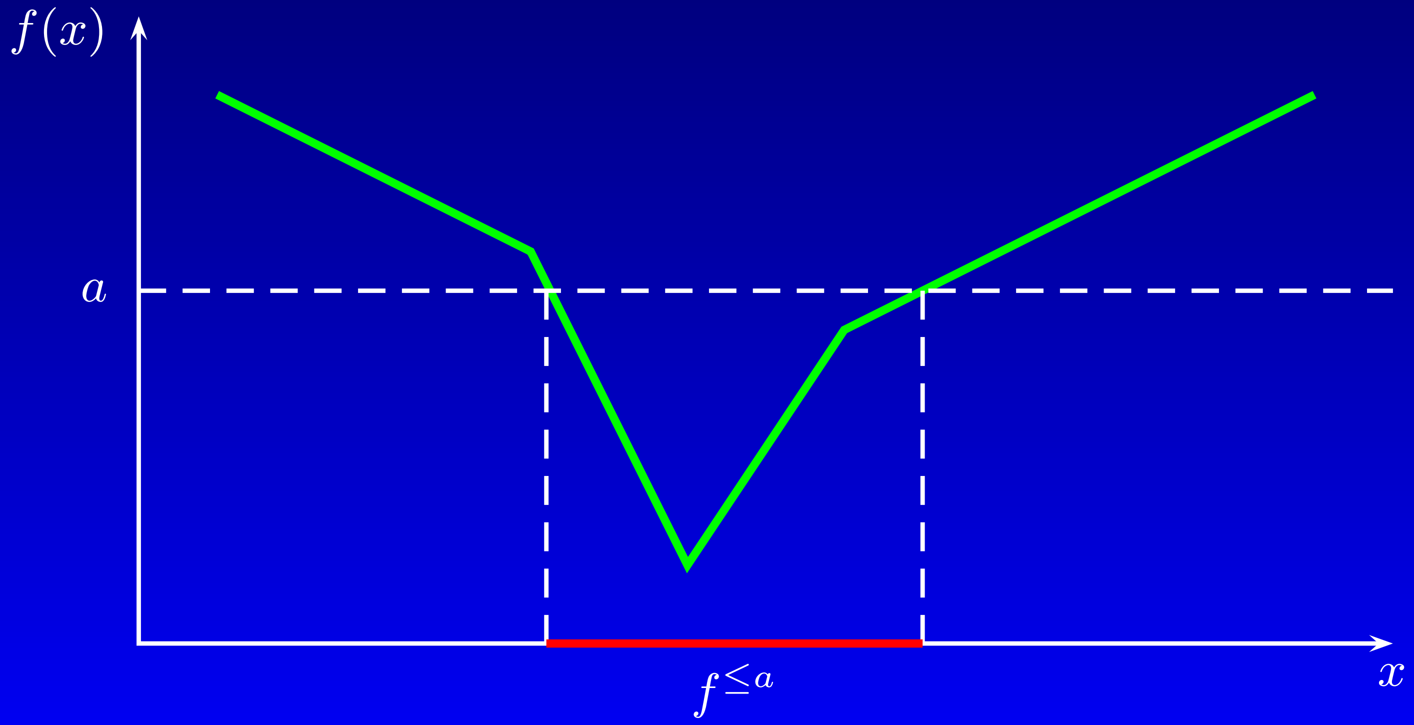
$$\lambda(\alpha) := \max_b \lambda^b(\alpha)$$

Quasiconvex Functions

Def: A function $f : D \to \mathbb{R}$, with $D \subseteq \mathbb{R}^d$ convex, is quasiconvex if the set

$$f^{\leq a} := \{x \in D : f(x) \leq a\}$$

is convex for any $a \in \mathbb{R}$



Quasiconvex Functions

Thr [Eppstein'01]: Function $\lambda(\alpha)$, $\alpha \in \mathbb{R}^d$, is quasiconvex

Prf:

- Since the max of a finite number of quasiconvex functions is quasiconvex, it is sufficient to show that each $\lambda^b(\alpha)$ is quasiconvex

- $\lambda^b(\alpha)$ is the positive root of $f^b(\lambda, \alpha) = 1 - \sum_j \lambda^{-\sum_i \alpha_i \delta^b_{i,j}}$

- Hence
$\lambda^{b,\leq a} = \{\alpha \in \mathbb{R}^d : \lambda^b(\alpha) \leq a\} = \{\alpha \in \mathbb{R}^d : \sum_j a^{-\sum_i \alpha_i \delta^b_{i,j}} \leq 1\}$

- $g^b(\alpha) := \sum_j a^{-\sum_i \alpha_i \delta^b_{i,j}}$ is convex as sum of convex functions, and trivially its level sets are convex, including $g^{b,\leq 1}$

Cor: Function $\lambda(\alpha)$ is quasiconvex over any convex $D \subseteq \mathbb{R}^d$

Applications to M&C

- We can use these facts to optimize the weights much faster in the Measure & Conquer framework

- Suppose we define a set of linear constraints on the weights such that

(a) the size of each subproblem does not increase

(b) the initial measure $m = m(\alpha)$ is upper bounded by n, where n is a *standard* measure for the problem

- This gives a convex domain of weights α. On that domain we can compute the minimum value $\lambda(\tilde{\alpha})$ of the quasiconvex function $\lambda(\alpha)$

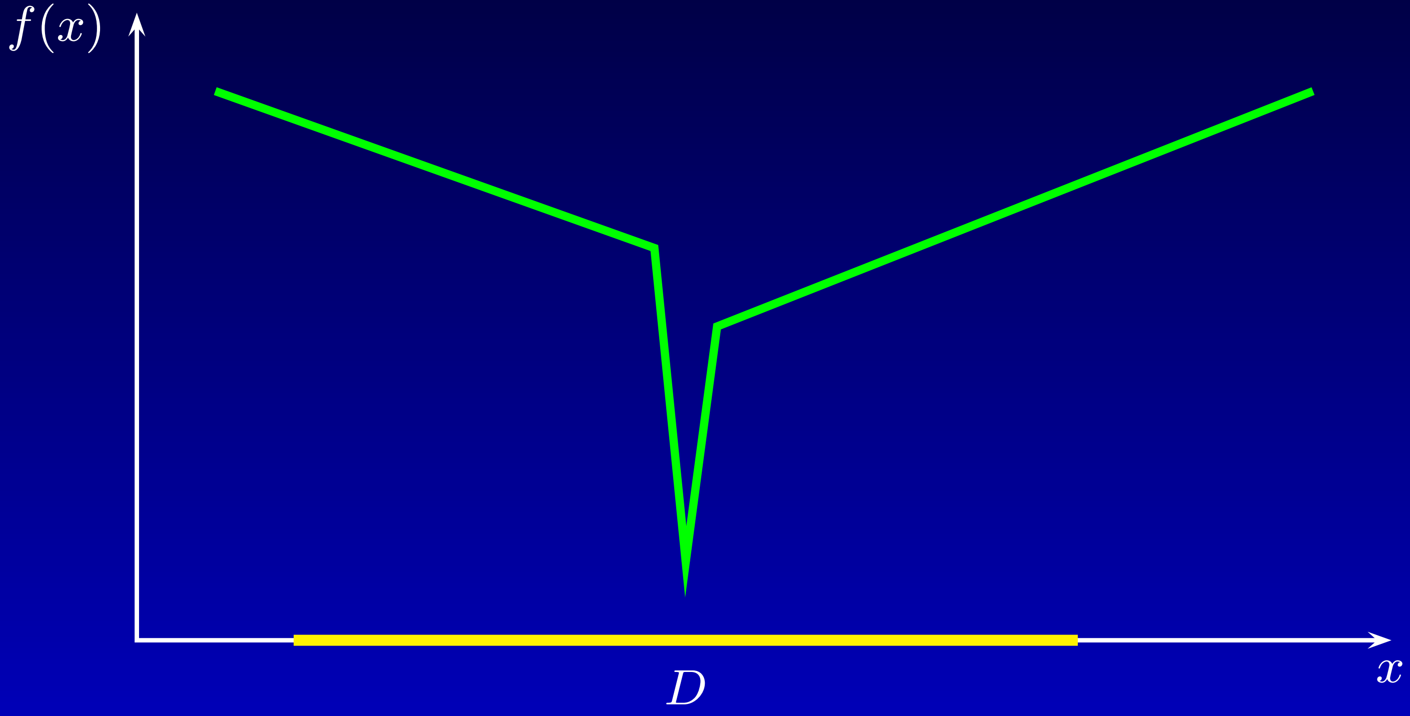
- The resulting running time is $O^*(\lambda(\tilde{\alpha})^{m(\tilde{\alpha})}) = O^*(\lambda(\tilde{\alpha})^n)$

Randomized Local Search

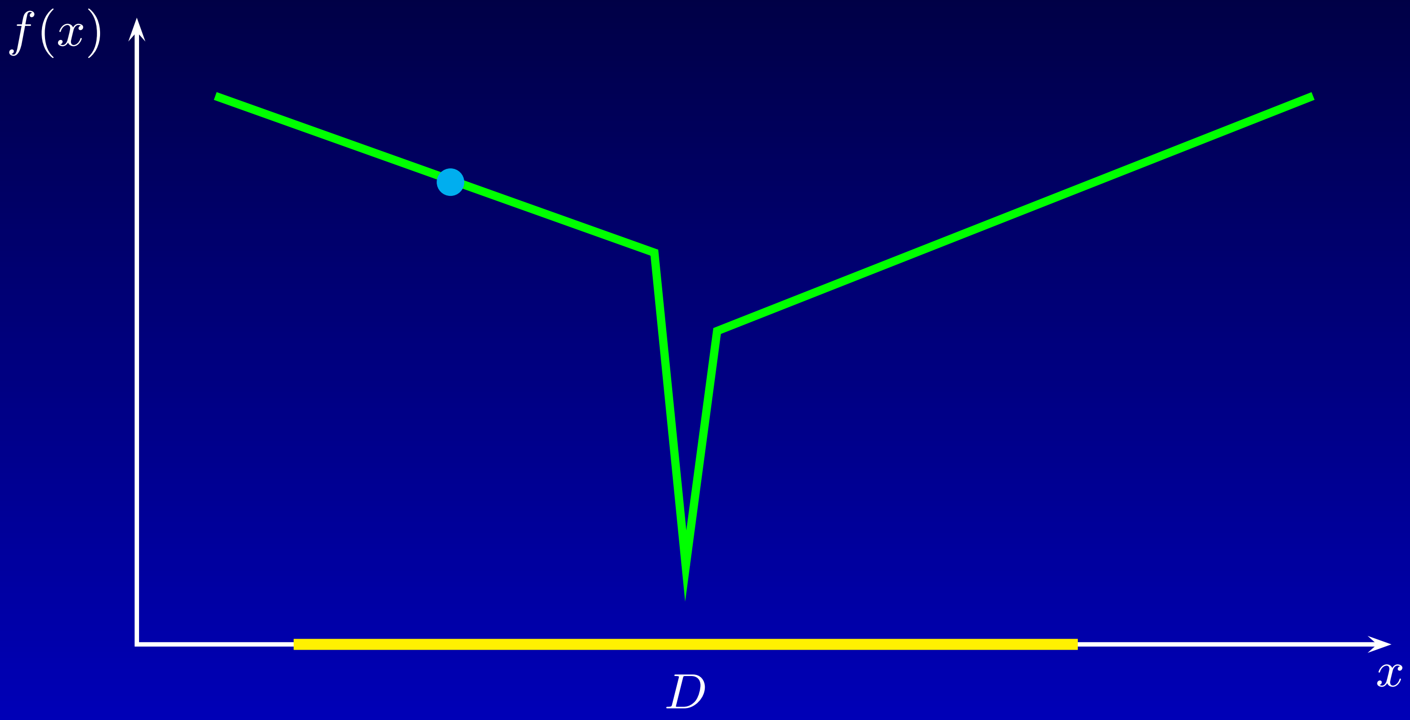
- There are known techniques to find efficiently the minimum of a quasi-convex functions (see e.g. [Eppstein'01,Gaspers])

- We successfully applied the following, very fast and easy to implement, approach based on *randomized local search* (in simulated annealing style)
  - ◇ We start from any feasible initial value α
  - ◇ We add to it a random vector in a given range $[-\Delta, \Delta]^d$
  - ◇ If the resulting α' is feasible and gives $\lambda(\alpha') \leq \lambda(\alpha)$, we set $\alpha = \alpha'$
  - ◇ We iterate the process, reducing the value of Δ if no improvement is achieved for a large number of steps
  - ◇ The process halts when Δ drops below a given value Δ'

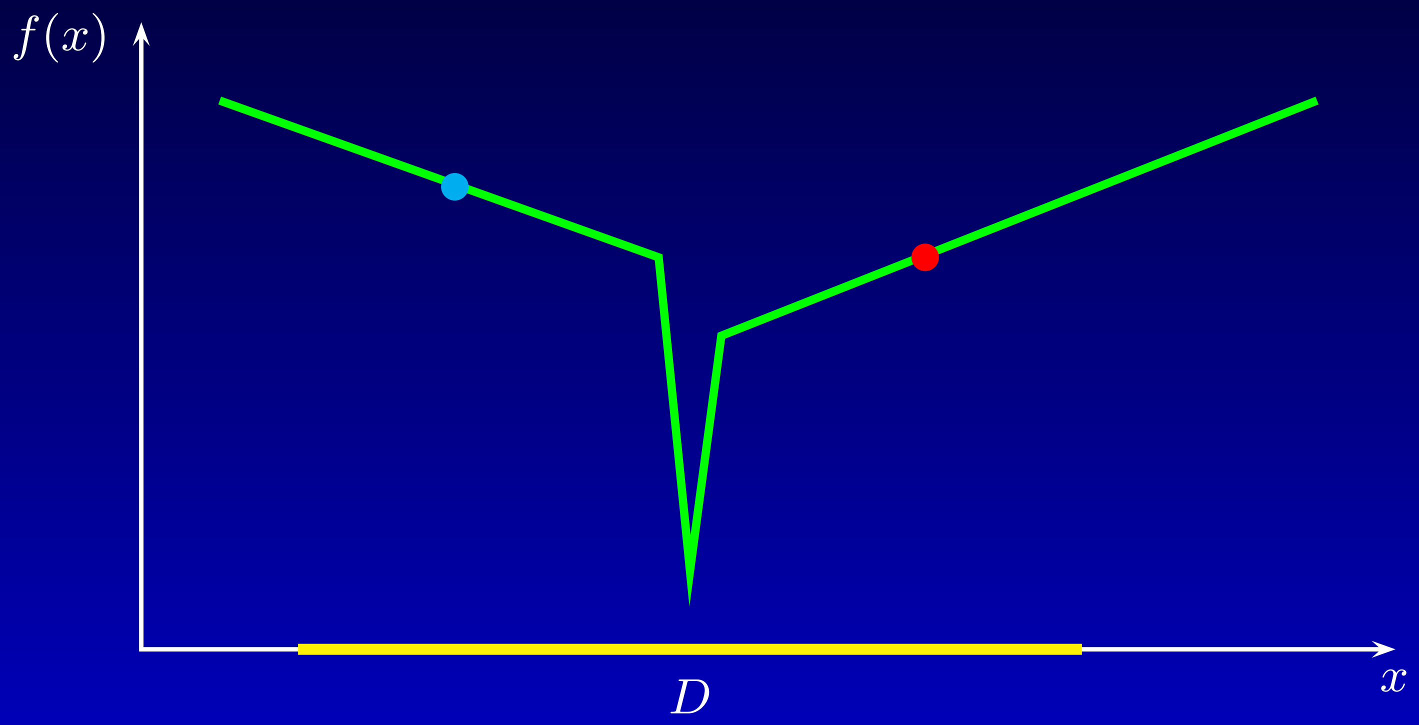
Randomized Local Search



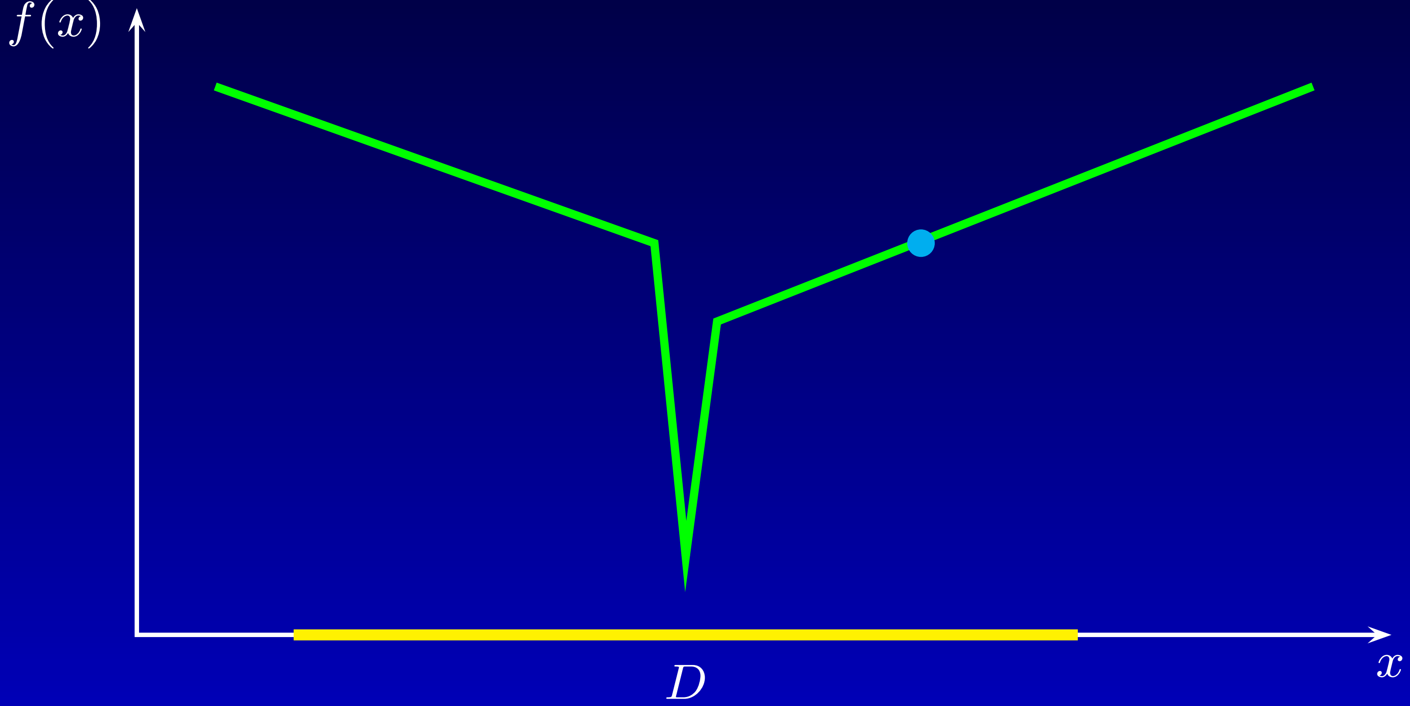
Randomized Local Search



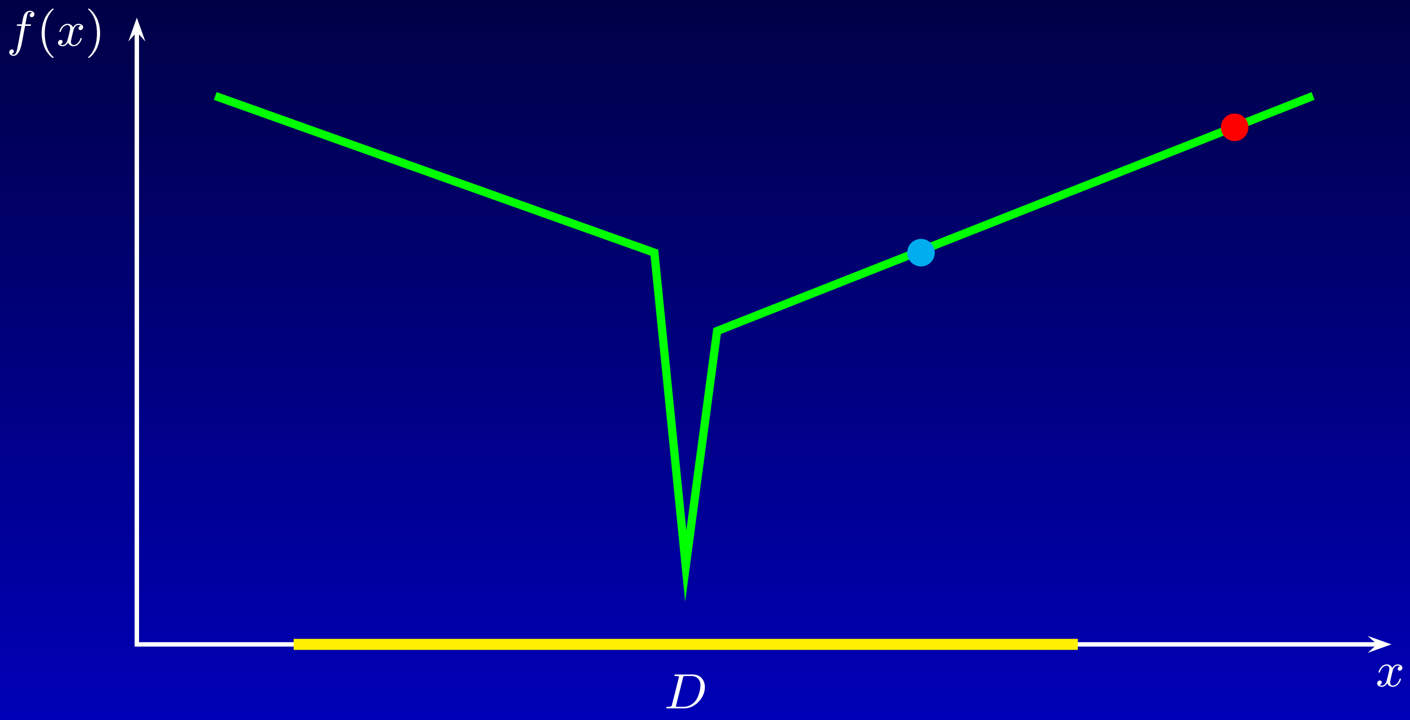
Randomized Local Search



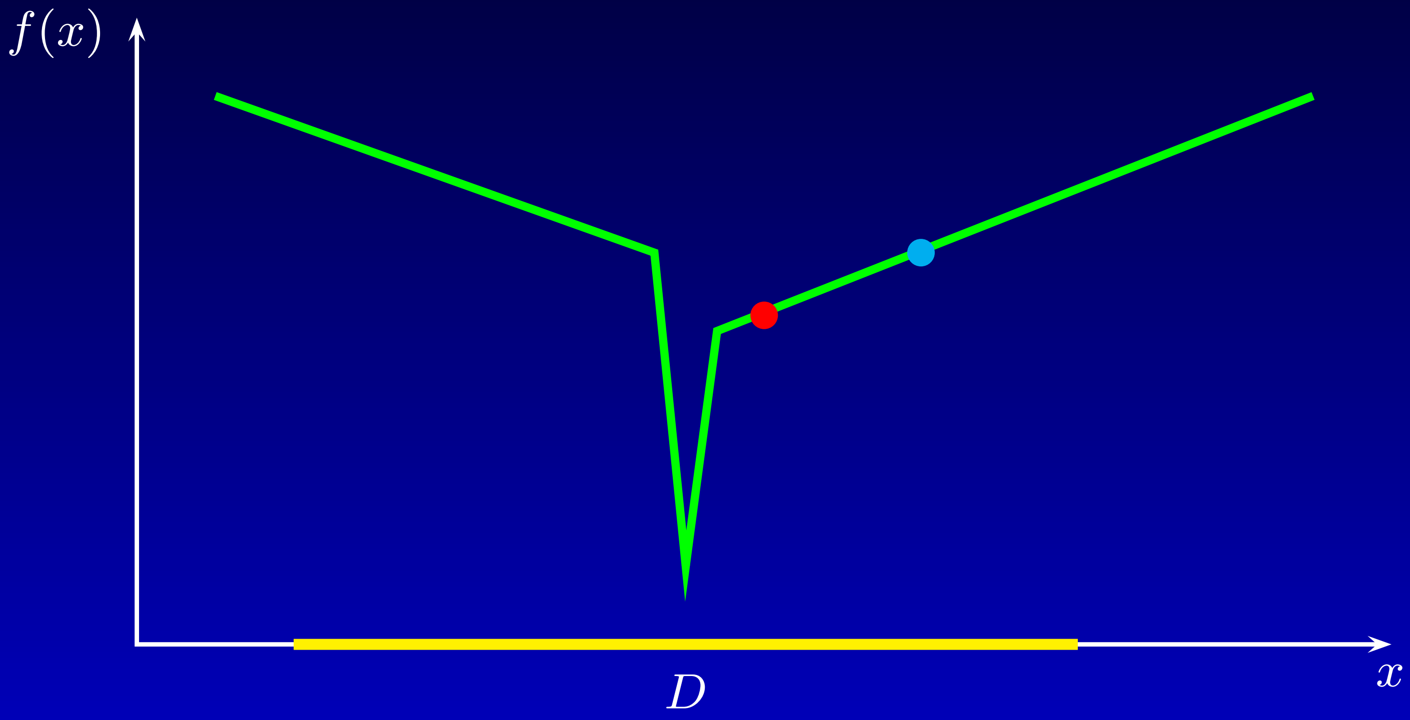
Randomized Local Search



Randomized Local Search



Randomized Local Search



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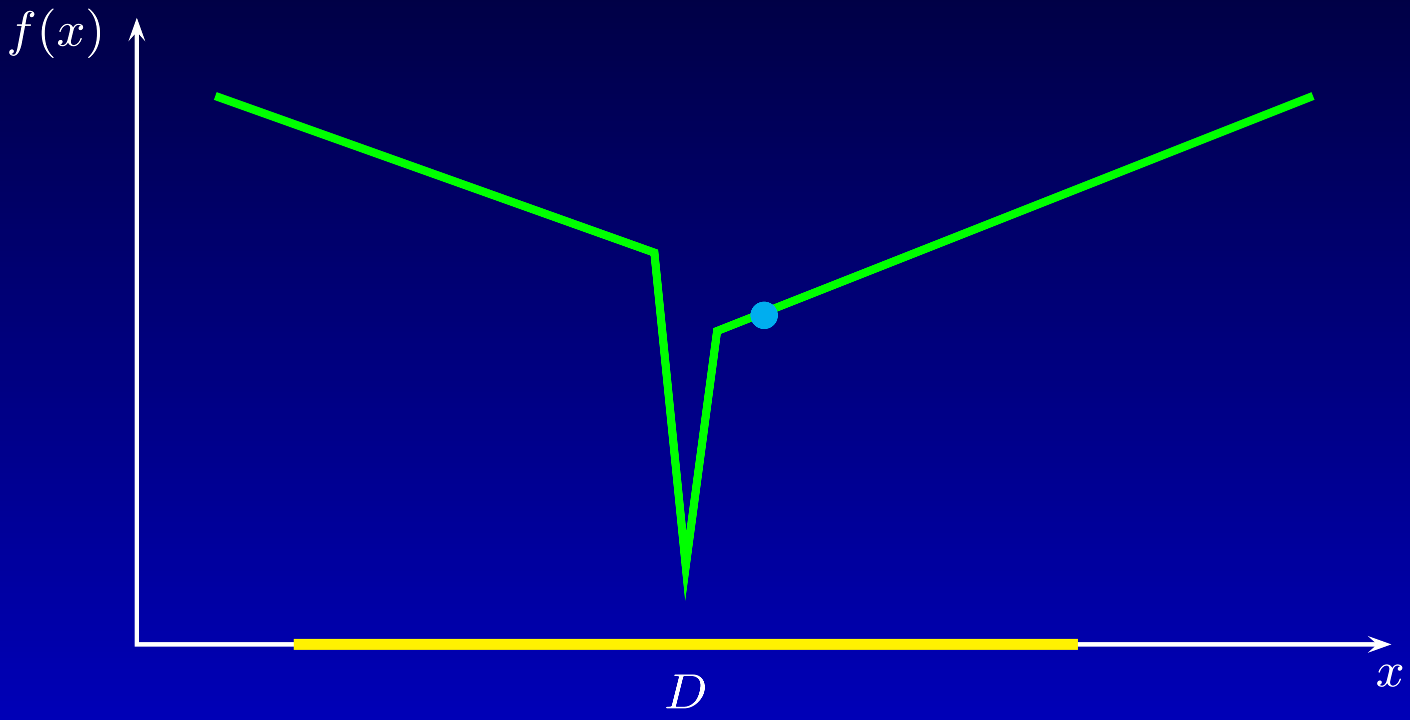
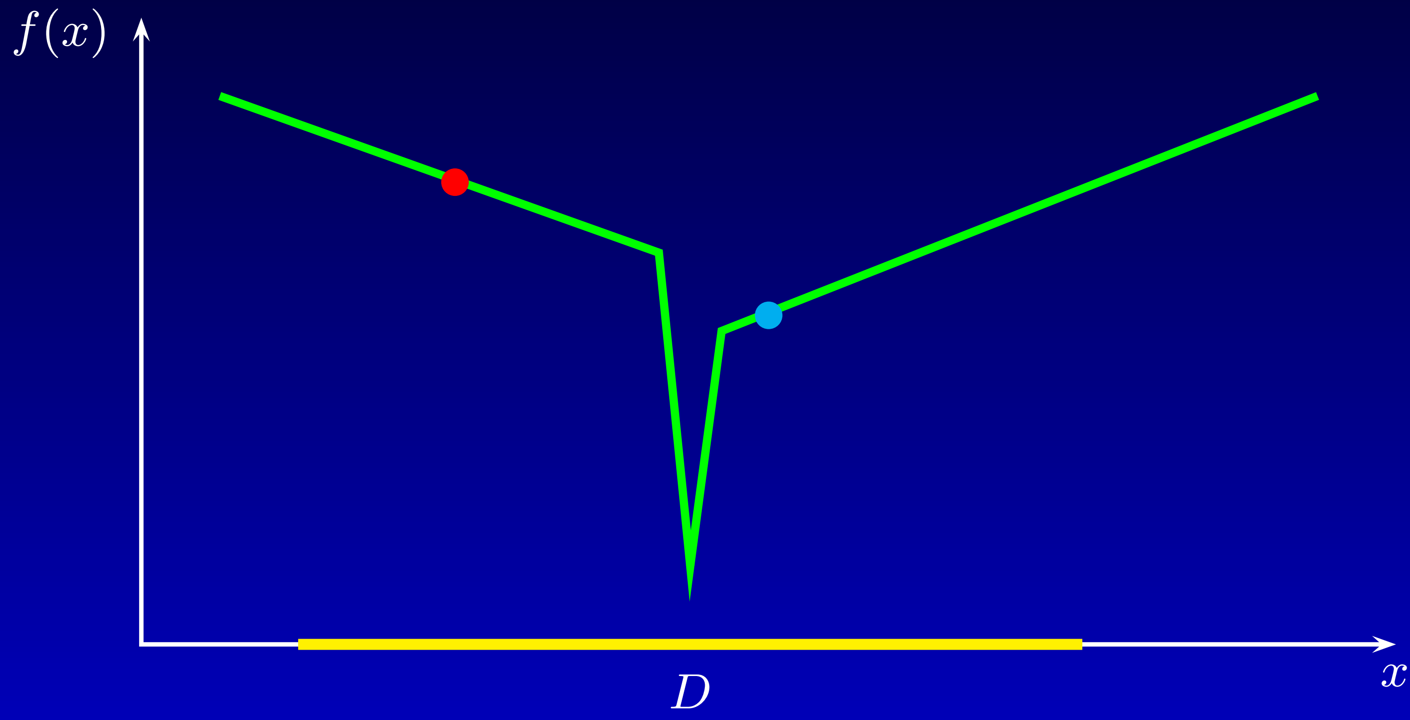
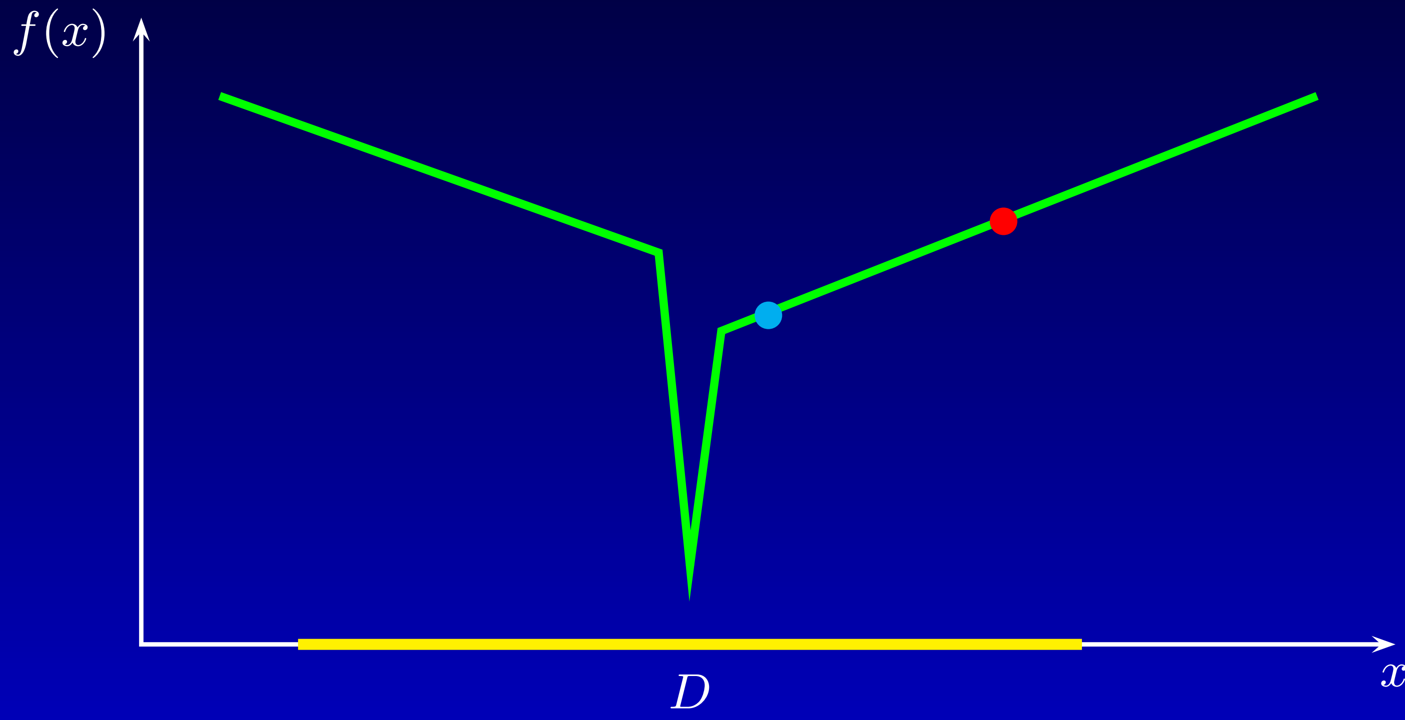


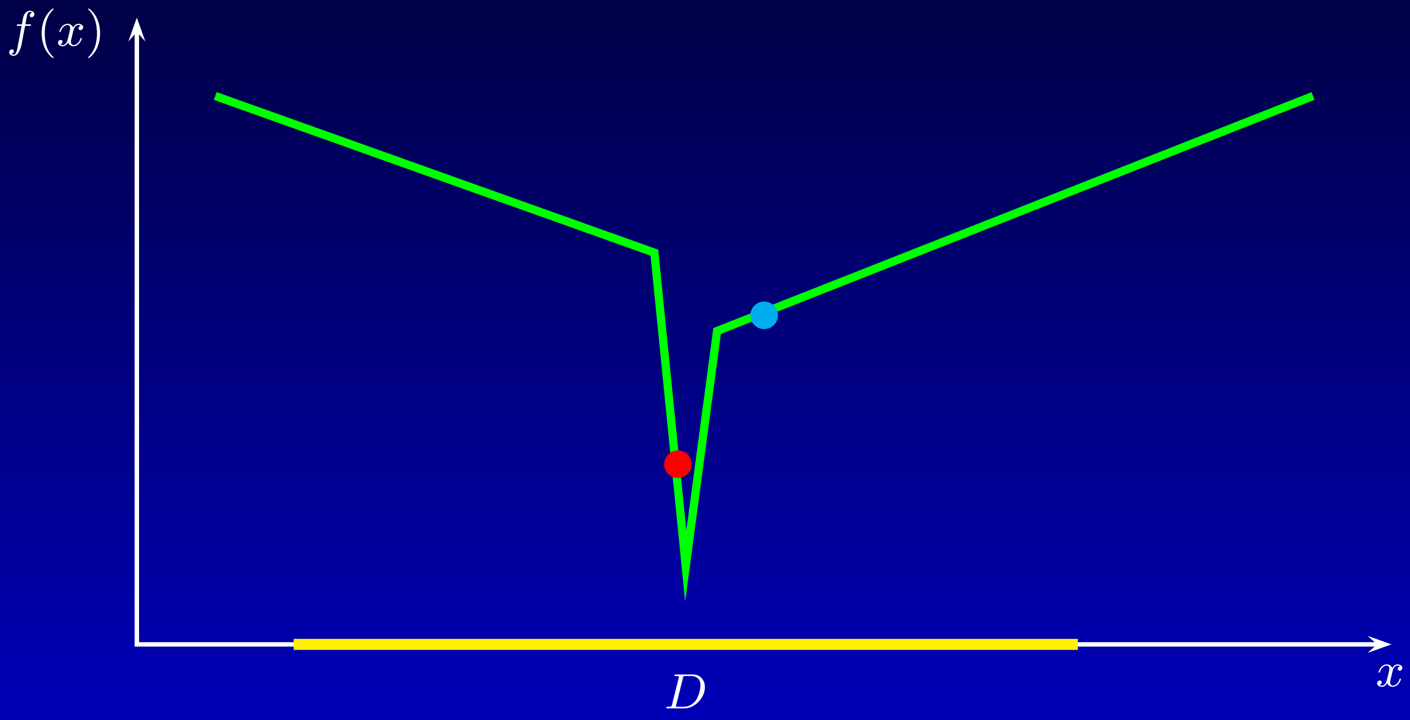
Randomized Local Search



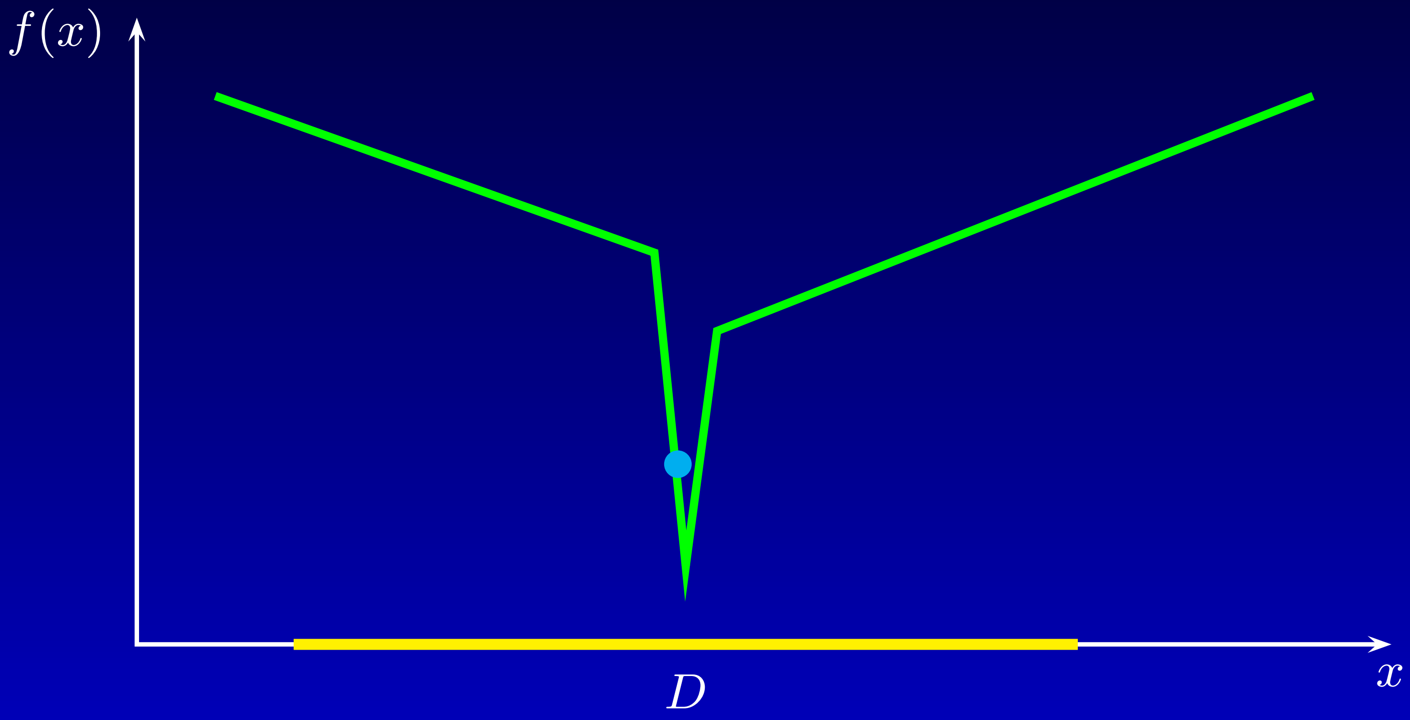
Randomized Local Search



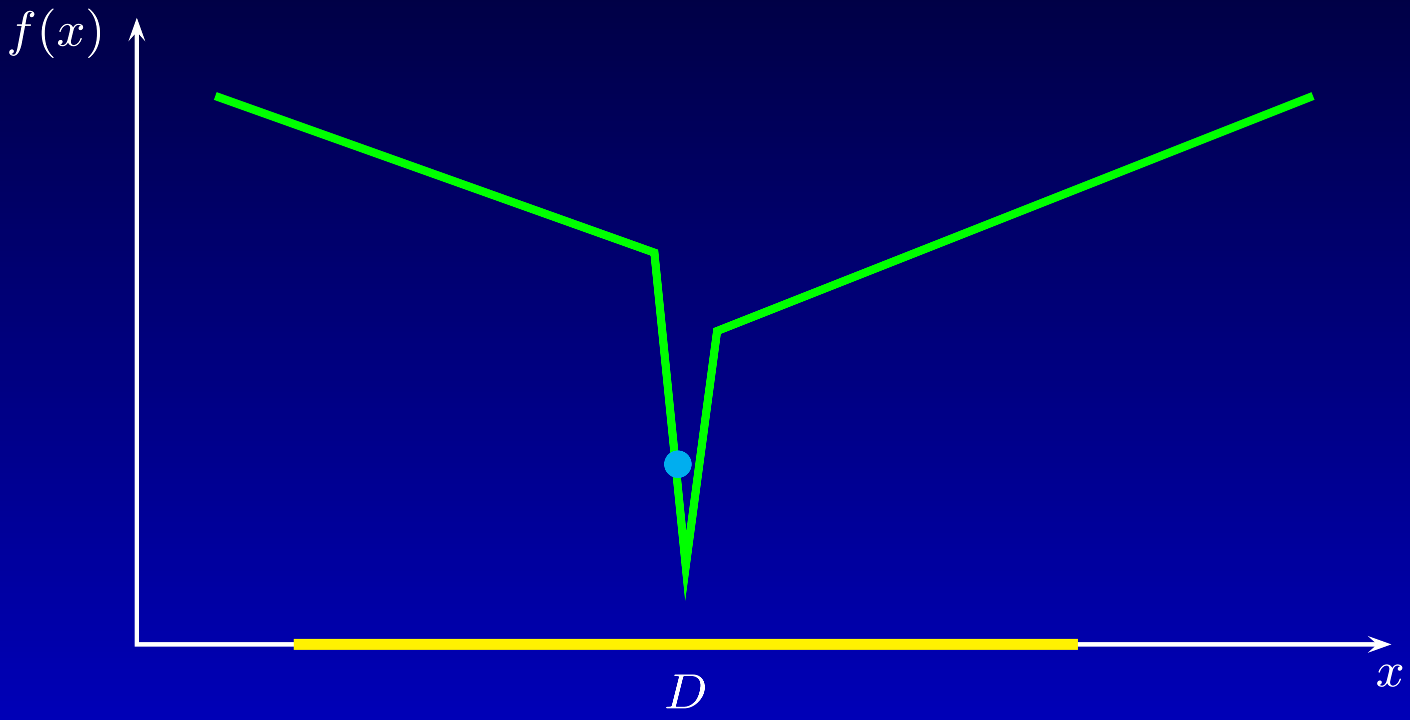
Randomized Local Search



Randomized Local Search



Randomized Local Search



Rem: This algorithm does not guarantee closeness to the optimal $\lambda(\tilde{\alpha})$. However it is accurate in practice. More important, it provides *feasible* upper bounds

Lower Bounds

Lower Bounds

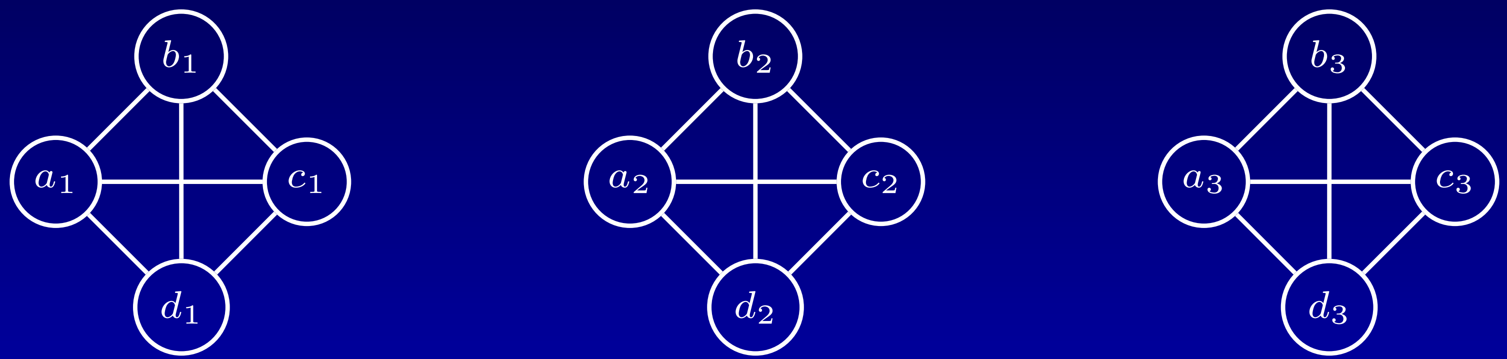
- Measure & Conquer sometimes leads to much better running time bounds
- Still, these bounds might not be tight
- Hence, it makes sense to search for (exponential) lower bounds on the running time of the algorithm considered (not of the problem!)
- A lower bound may give an idea of how far the analysis is from being tight

A Lower Bound for `mis`

Thr 4: The running time of `mis` is $\Omega(2^{n/4})$

Prf:

- Consider the graph G_k consisting of $k = n/4$ copies of a K_4



- The algorithm might branch at a_1. In both subproblems $\{a_1, b_1, c_1, d_1\}$ is removed, either immediately or later on by folding. This leaves a G_{k-1}

- We obtain a recurrence of the type $P(n) \geq 2P(n-4)$ for the number of subproblems, which gives $P(n) \geq 2^{n/4}$

A Lower Bound for `mis`

Thr 4: The running time of `mis` is $\Omega(2^{n/4})$

Exr 8: Find a larger lower bound on the running time of `mis` (Hint: $\Omega(3^{n/6}) = \Omega(1.20^n)$, maybe better)

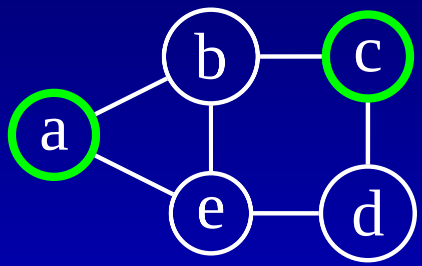
Exr 9*: Consider the variant of `mis` where the algorithm, after the base case, branches on connected components when possible. Can you find a good lower bound on the running time of this modified algorithm?

Rem: Typically finding lower bounds on connected graphs is much more complicated

Applications of Measure & Conquer

Independent Set

Def: Given $G = (V, E)$, the *independent set* problem (MIS) is to determine the maximum cardinality $\alpha(G)$ of a subset of pairwise non-adjacent nodes (*independent set*)



$$\alpha(G) = 2$$

Independent Set

Thr [Fomin,Grandoni,Kratsch'06-'09]: MIS can be solved in $O^*(1.221^n)$ time and polynomial space

Prf:

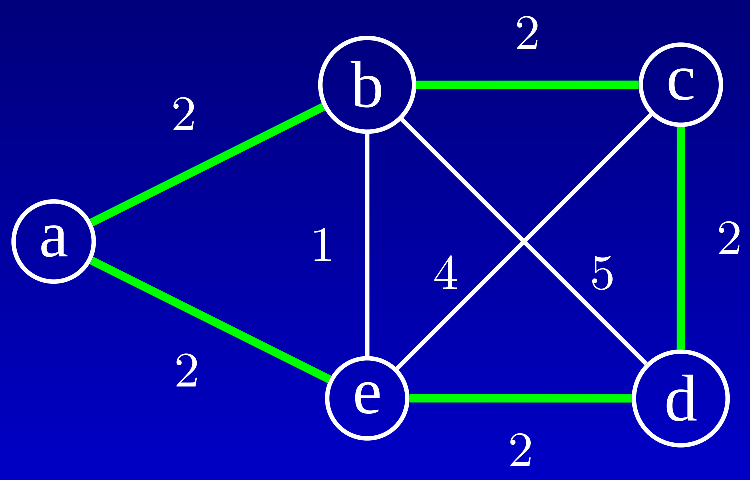
- Simple branching algorithm

int `mis`(G) {
  if($|V(G)| \leq 1$) **return** $|V(G)|$;
  if($\exists$ component $C \subset G$) **return** `mis`(C)+`mis`($G - C$);
  if($\exists$ vertices v and w: $N[w] \subseteq N[v]$) **return** `mis`($G - \{v\}$);
  if($\exists$ a vertex v, with $d(v) = 2$) **return** 1+`mis`(G_v);
  select a vertex v of maximum degree, which minimizes $|E(N(v))|$;
  return $\max\{$`mis`$(G - \{v\} - M(v))$, 1+`mis`$(G - N[v])\}$;
}

- Analysis similar to the one outlined before

Traveling Salesman Problem

Def: Given a weighted $G = (V, E)$, the *traveling salesman problem* problem (TSP) is to compute a minimum weight cycle spanning V (*TSP tour*)



Traveling Salesman Problem

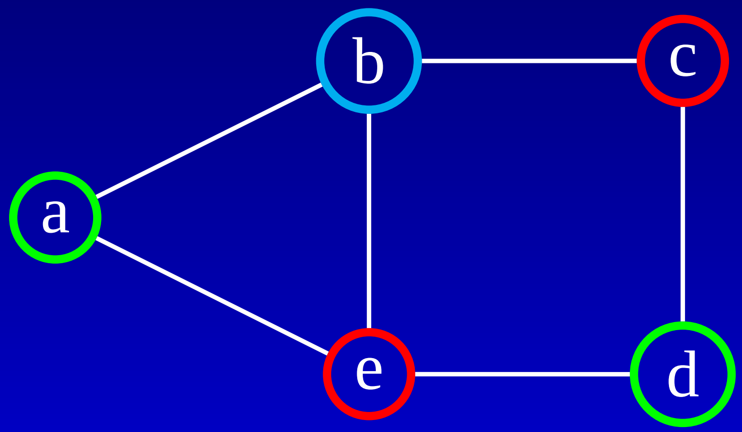
Thr [Eppstein'03-'07]: TSP can be solved in $O^*(1.260^n)$ time in cubic graphs

Prf:

- Design a non-trivial branching algorithm
- Analyze it using, as measure, $|V| - |F| - |C| \leq |V|$
- Here F is a set of *forced* edges and C the set of 4-cycles of G which induce connected components in $G - F$

3-Coloring

Def: Given $G = (V, E)$ and a set of 3 colors, the 3-*coloring* problem (3-COL) is to find an assignment of colors to nodes such that adjacent nodes are colored differently



3-Coloring

Def: Given a set of variables on domains of size $\leq a$ and a set of constraints each one involving at most b variables, the (a, b)-*constraint satisfaction* problem (CSP) is to find an assignment of the variables satisfying all the constraints

Rem: 3-COL is a special case of $(3, 2)$-CSP

3-Coloring

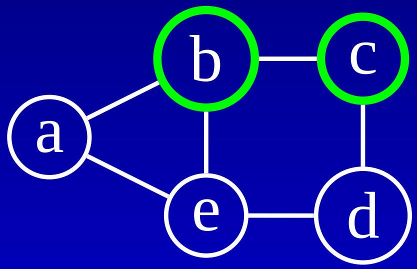
Thr [Beigel,Eppstein'00-'05]: 3-COL can be solved in $O^*(1.329^n)$ time

Prf:

- Non-trivial reduction to $(3, 2)$-CSP
- Non-trivial branching algorithm solving $(4, 2)$-CSP in $O^*(1.365^n)$ time
- In the analysis the measure is a linear combination $n_3 + \alpha\, n_4$ of the number of variables with domain of size 3 and 4 (variables with smaller domain can be filtered out)

Dominating Set

Def: Given $G = (V, E)$, the *dominating set* problem (MDS) is to determine the minimum cardinality $\delta(G)$ of a subset of nodes D such that any node in $V - D$ is adjacent to some node in D (*dominating set*)

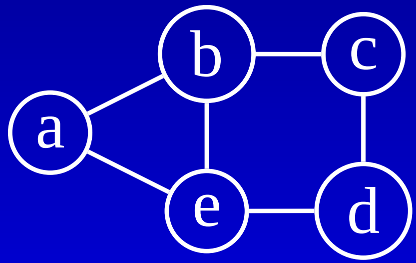


$$\delta(G) = 2$$

Dominating Set

Def: Given a universe $\mathcal{U}$ and a collection of subsets $\mathcal{S} \subseteq 2^{\mathcal{U}}$, the *set cover* problem (MSC) is to determine a minimum cardinality subcollection $\mathcal{C} \subseteq \mathcal{S}$ such that $\cup_{S \in \mathcal{C}} S = \mathcal{U}$ (*set cover*)

Rem: MDS can reduced to MSC by letting $\mathcal{U} = V$ and $\mathcal{S} = \{N[v] : v \in V\}$. This instance has n subsets and n elements



$$\mathcal{U} = \{a, b, c, d, e\}$$

$$\begin{aligned} S_a &= \{a, b, e\} \\ S_b &= \{a, b, c, e\} \\ S_c &= \{b, c, d\} \\ S_d &= \{c, d, e\} \\ S_e &= \{a, b, d, e\} \end{aligned}$$

Dominating Set

Thr [Grandoni'04-'06]: MDS can be solved in $O^*(1.803^n)$ time

Proof: Design a simple algorithm solving MSC in $O^*(1.381^{|\mathcal{U}|+|\mathcal{S}|})$ time $\Rightarrow O^*(1.381^{2n})$ time algo for MDS

int msc($\mathcal{S}$) {
  if($|\mathcal{S}| = 0$) **return** 0;
  if($\exists S, R \in \mathcal{S} : S \subseteq R$) **return** msc($\mathcal{S}\backslash\{S\}$);
  if($\exists u \in \mathcal{U}(\mathcal{S})\exists$ a unique $S \in \mathcal{S} : u \in S$) **return** $1+$msc(del(S, $\mathcal{S}$));
  take $S \in \mathcal{S}$ of maximum cardinality;
  if($|S| = 2$) **return** poly-msc($\mathcal{S}$)
  return $\min\{$msc($\mathcal{S}\backslash\{S\}$), $1+$msc(del(S, $\mathcal{S}$))$\}$;
}

Exr 10: Prove the theorem above

Dominating Set

Thr [Fomin,Grandoni,Kratsch'05-'09]: MDS can be solved in $O^*(1.527^n)$ time

Proof:

- Consider the same reduction to MSC and the same algorithm as before

- Give a different weight to sets of different cardinality and to elements of different frequency

Exr 11*: Prove the theorem above

Thr [van Rooij,Bodlaender'08]: MDS can be solved in $O^*(1.507^n)$ time

Variants of Dominating Set

Def: Given $G = (V, E)$, the *minimum independent dominating set* problem (MIDS) is to determine the minimum cardinality of a dominating set of G which is also an independent set

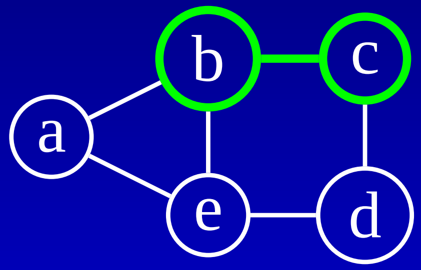
Thr [Gasper,Liedloff'06]: MIDS can be solved in $O^*(1.358^n)$ time

Def: Given $G = (V, E)$, the *minimum dominating clique* problem (MDC) is to determine the minimum cardinality of a dominating set of G which is also a clique

Thr [Kratsch,Liedloff'07]: MDC can be solved in $O^*(1.324^n)$ time

Connected Dominating Set

Def: Given $G = (V, E)$, the *connected dominating set* problem (ConDomS) is to determine the minimum cardinality $\delta'(G)$ of a dominating set of G which induces a connected graph (*connected dominating set*)



$$\delta'(G) = 2$$

Connected Dominating Set

Thr [Fomin,Grandoni,Kratsch'06-'08]: Connected dominating set can be solved in $O^*(1.941^n)$ time

Proof:

- Design an algorithm which gradually expands a connected graph, until it becomes dominating

- Assign a different weight to nodes dominating a different number of nodes not yet dominated

- Assign an extra weight to nodes which are still not selected nor discarded, giving a smaller extra weight to nodes whose removal makes the problem infeasible

Rem: without the refined measure one does not improve on trivial 2^n!

Combinatorial Bounds via M&C

- M&C can be used to derive better combinatorial bounds

Thr [Fomin,Grandoni,Pyatkin,Stepanov'05-'08]: An n-node graph has $O^*(1.716^n)$ minimal dominating sets

Prf: Design a listing algorithm and analyze it via M&C

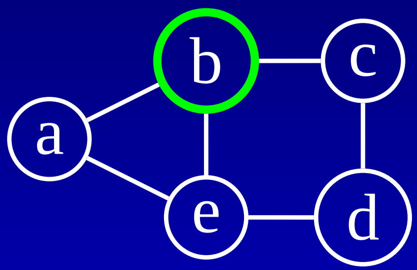
- Listing algorithms can often be used to solve weighted problems, where reduction rules are harder to get

Thr [Fomin,Grandoni,Pyatkin,Stepanov'05]: The weighted minimum dominating set problem can be solved in $O^*(1.578^n)$ time

Prf: Use a variant of the listing algorithm above, implementing a trivial weighted set cover reduction rule

Feedback Vertex Set

Def: Given $G = (V, E)$, the *feedback vertex set* problem (FVS) is to determine the minimum cardinality $\phi(G)$ of a subset of nodes whose removal makes G acyclic (*feedback vertex set*)



$$\phi(G) = 1$$

Feedback Vertex Set

Thr [Razgon+Fomin,Gaspers,Pyatkin'06-'08]: FVS can be solved in $O^*(1.755^n)$ time

Prf:

- Design an algorithm based on branching rules and maximum independent sets computation to solve the equivalent maximum induced forest problem
- Analyze it using, as measure,

$$0 \cdot |F| + 1 \cdot |N(t)| + (1+\alpha)|V - F - N(t)|$$

- Here F is a set of forced nodes and t is an *active* node

Apologies

I apologize for related and improved results that I forgot to mention

THANKS!!!